SECOND-ORDER OPTIMALITY CONDITIONS FOR THE BOLZA PROBLEM WITH PATH CONSTRAINTS

L. J. Wood
California Institute of Technology

Abstract

A set of sufficient conditions for a weak minimum is derived for a form of the nonsingular Bolza problem of variational calculus, with interior point constraints and discontinuities in the system equations. Generalized versions of the conjugate point/focal point, normality, convexity and nontangency conditions associated with the ordinary Bolza problem are obtained. The resulting set of sufficient conditions is minimal, in that only minor modifications are required in order to obtain necessary conditions for normal, nonsingular problems of this form. These conditions are relatively easy to implement. Analogous second-order optimality conditions for problems with natural corners or control constraints are also obtained. Previously stated sufficiency conditions for problems with control constraints are shown to be unnecessarily restrictive, in some cases.

Introduction

Many optimization problems for deterministic, nonlinear, time-varying systems are expressible as a form of the variational problem known as the problem of Bolza [1]. The form of the Bolza problem considered in this paper may be expressed in modern control notation as follows: Among the set of all continuous n-dimensional state variable functions x(t) and piecewise continuous m-dimensional control variable functions u(t) satisfying differential equations of the form

$$\dot{x}(t) = f^{(i)}(x,u,t), \quad t_{i-1}^{+} \le t \le t_i^{-}, \quad i = 1 \ldots, N, \tag{1}$$

and endpoint and interior point constraints of the form

$$t_o, \; x(t_o) \text{ given}, \tag{2}$$

$$Y^{(i)}[x(t_i), t_i] - y^{(i)} = 0, \quad i = 1, \ldots, N, \tag{3}$$

find the set that will minimize the scalar performance index

$$J = \sum_{i=1}^{N} g^{(i)}[x(t_i), t_i] + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} L^{(i)}(x,u,t)dt , \tag{4}$$

where $t_o < t_1 < \ldots < t_N$. The t_i, i=1, ..., N, are assumed to be unspecified. u(t) is continuous except at these points. $Y^{(i)}$ and $y^{(i)}$ are vectors of dimension $r_i \le n$, i=1, ..., N. The functions $f^{(i)}$, $L^{(i)}$, $Y^{(i)}$, and $g^{(i)}$ are assumed to be twice continuously differentiable with respect to their arguments. A dot above a variable denotes differentiation with respect to time.

The ordinary Bolza problem, as considered by Bliss [1], has a continuous integrand and continuous system equations and has no interior point constraints and no functions $g^{(i)}(x_i, t_i)$, i=1, ..., N-1, appearing in the performance index. Denbow [2] was the first to consider this type of generalized Bolza problem extensively. He described a transformation by which this problem can be expressed as an ordinary Bolza problem. This transformation involves scaling the N time intervals $t_i - t_{i-1}$ to a common length, and increasing the dimension of the state from n to Nn.

First-order necessary conditions for a local minimum in J were deduced by Denbow using such an approach. Equivalent conditions have been obtained far more simply in [3, 4]. Trajectories which satisfy these conditions are only candidates for optimality, however. Use of the second variation is generally needed to determine whether such trajectories are, in fact, locally minimizing.

Bliss [1] has stated sets of necessary and sufficient conditions for positive definiteness of the second variation in the ordinary Bolza problem, but these conditions are difficult to apply to realistic optimization problems. More easily implemented second-order optimality conditions are stated in [4, 5]. Several errors in [4, 5] have been noted, and the results extended to more general problems, in [6, 7]. Speyer [8] has extended the numerical technique of McReynolds [5] for solving two-point boundary value problems to problems with interior point constraints, and has derived expressions for neighboring optimal feedback gains, but he did not consider second-order optimality conditions extensively. In this paper, the results of [6, 7] are extended to the generalized Bolza problem stated above.

With suitable modifications, these results can be applied to problems with natural corners or with control constraints. The resulting sufficiency conditions for problems with control constraints are less restrictive than those in [9], in some cases.

Stationary Trajectories

The state equations (1) and interior and terminal constraints (3) may be adjoined to the performance index (4) with the use of the Lagrange multipliers $\lambda(t)$ and $\nu^{(i)}$, as follows:

$$\overline{J} = \sum_{i=1}^{N} \Big\{ G^{(i)}(x_i, t_i, \nu^{(i)}) - \nu^{(i)T} y^{(i)} + \int_{t_{i-1}}^{t_i} [H^{(i)}(x, u, \lambda, t) - \lambda^T \dot{x}]\, dt \Big\} \tag{5}$$

where $G^{(i)} \triangleq g^{(i)} + \nu^{(i)T} Y^{(i)}$, $H^{(i)} \triangleq L^{(i)} + \lambda^T f^{(i)}$ (6,7)

First-order necessary conditions for a local minimum of J are known to be [3,4] Eqs. (1-3) and

$$\dot{\lambda}^T = -H_x^{(i)}, \quad t_{i-1}^+ \le t \le t_i^- \tag{8}$$

$$0 = H_u^{(i)}, \quad t_{i-1}^+ \le t \le t_i^- \tag{9}$$

$$\lambda^T(t_i^-) = \lambda^T(t_i^+) + G_{x_i}^{(i)}, \quad i = 1, \ldots, N-1 \tag{10}$$

$$\begin{aligned} z^{(i)}(x_i, u_{i-}, u_{i+}, \nu^{(i)}, \lambda_{i+}, t_i) &\triangleq G_{t_i}^{(i)} + L^{(i)}(t_i^-) \\ &\quad - L^{(i+1)}(t_i^+) + (G_{x_i}^{(i)} + \lambda^T(t_i^+)) f^{(i)}(t_i^-) \\ &\quad - \lambda^T(t_i^+) f^{(i+1)}(t_i^+) = 0, \quad i = 1, \ldots, N-1 \end{aligned} \tag{11}$$

$$\lambda^T(t_N) = G_{x_N}^{(N)} \tag{12}$$

$$z^{(N)}(x_N, u_N, \nu^{(N)}, t_N) \triangleq G_{t_N}^{(N)} + G_{x_N}^{(N)} f^{(N)}(t_N) + L^{(N)}(t_N) = 0 \tag{13}$$

The quantities $H_x^{(i)}$, $H_u^{(i)}$, etc., denote partial derivatives. If x and u are taken to be column vectors, then $H_x^{(i)}$ and $H_u^{(i)}$ are row vectors. Quantities discontinuous at t_i have left hand and right hand limits denoted by $x(t_i^-)$ and $x(t_i^+)$, etc. The subscript i denotes evaluation of a quantity at t_i.

Neighboring Stationary Trajectories

By linearizing Eqs. (1-3) and (8-13), the following equations governing perturbations about a stationary trajectory may be obtained:

$$\delta\dot{x} = f_x^{(i)} \delta x + f_u^{(i)} \delta u, \quad t_{i-1}^+ \le t \le t_i^- \tag{14}$$

$$\delta\dot{\lambda} = -H_{xx}^{(i)} \delta x - H_{xu}^{(i)} \delta u - f_x^{(i)T} \delta\lambda, \quad t_{i-1}^+ \le t \le t_i^- \tag{15}$$

$$0 = H_{ux}^{(i)} \delta x + H_{uu}^{(i)} \delta u + f_u^{(i)T} \delta\lambda, \quad t_{i-1}^+ \le t \le t_i^- \tag{16}$$

$$t_o, \ \delta x(t_o) \text{ specified} \tag{17}$$

$$\begin{bmatrix} \delta\lambda(t_i^-) \\ dy^{(i)} \\ dz^{(i)} \end{bmatrix} = \begin{bmatrix} G_{x_i x_i}^{(i)} & Y_{x_i}^{(i)T} & z_{x_i}^{(i)T} \\ Y_{x_i}^{(i)} & 0 & dY^{(i)}/dt_i \\ z_{x_i}^{(i)} & (dY^{(i)}/dt_i)^T & dz^{(i)}/dt_i \end{bmatrix} \begin{bmatrix} \delta x(t_i^-) \\ d\nu^{(i)} \\ dt_i \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ z_{\lambda_{i+}}^{(i)} \end{bmatrix} \delta\lambda(t_i^+), \quad i = 1, \ldots, N-1 \tag{18}$$

$$\begin{bmatrix} \delta\lambda(t_N) \\ dy^{(N)} \\ dz^{(N)} \end{bmatrix} = \begin{bmatrix} G_{x_N x_N}^{(N)} & Y_{x_N}^{(N)T} & z_{x_N}^{(N)T} \\ Y_{x_N}^{(N)} & 0 & dY^{(N)}/dt_N \\ z_{x_N}^{(N)} & (dY^{(N)}/dt_N)^T & dz^{(N)}/dt_N \end{bmatrix} \begin{bmatrix} \delta x(t_N) \\ d\nu^{(N)} \\ dt_N \end{bmatrix} \tag{19}$$

where

$$dY^{(i)}/dt_i = Y_{t_i}^{(i)} + Y_{x_i}^{(i)} f^{(i)}(t_i^-), \quad i=1, \ldots, N \tag{20}$$

$$z_{x_i}^{(i)} = G_{t_i x_i}^{(i)} + f^{(i)}(t_i^-)^T G_{x_i x_i}^{(i)} + H_x^{(i)}(t_i^-) - H_x^{(i+1)}(t_i^+), \tag{21}$$

$$z_{\lambda_{i+}}^{(i)} = f^{(i)}(t_i^-)^T - f^{(i+1)}(t_i^+)^T, \tag{22}$$

$$\begin{aligned} dz^{(i)}/dt_i &= G_{t_i t_i}^{(i)} + f^{(i)}(t_i^-)^T G_{x_i t_i}^{(i)} + G_{t_i x_i}^{(i)} f^{(i)}(t_i^-) \\ &\quad + f^{(i)}(t_i^-)^T G_{x_i x_i}^{(i)} f^{(i)}(t_i^-) + \Big[H_t^{(i)} + H_x^{(i)} f^{(i)}\Big]_{t=t_i^-} \\ &\quad - \Big[H_t^{(i+1)} + H_x^{(i+1)} f^{(i+1)}\Big]_{t=t_i^+} - 2\,[f^{(i)}(t_i^-)^T \\ &\quad - f^{(i+1)}(t_i^+)^T]\, H_x^{(i+1)^T}(t_i^+), \quad i = 1, \ldots, N-1 \end{aligned} \tag{23}$$

$$z_{x_N}^{(N)} = G_{t_N x_N}^{(N)} + f^{(N)}(t_N)^T G_{x_N x_N}^{(N)} + H_x^{(N)}(t_N), \tag{24}$$

$$\begin{aligned} dz^{(N)}/dt_N &= \Big[G_{t_N t_N}^{(N)} + f^{(N)T} G_{x_N t_N}^{(N)} + G_{t_N x_N}^{(N)} f^{(N)} \\ &\quad + f^{(N)T} G_{x_N x_N}^{(N)} f^{(N)} + H_t^{(N)} + H_x^{(N)} f^{(N)}\Big]_{t=t_N} \end{aligned} \tag{25}$$

The above derivatives of $z^{(i)}$ have been written out explicitly, since they are sometimes evaluated incorrectly. Quantities such as $H_{xu} \triangleq \partial(H_x^T)/\partial u$ are matrices of second partial derivatives.

A Backward Sweep Solution of the Linearized Variational Equations

If $H_{uu}^{(i)}(t)$ is nonsingular, Eq. (16) may be solved for $\delta u(t)$, and $\delta u(t)$ may be eliminated from Eqs. (14) and (15), yielding

$$\delta u(t) = -H_{uu}^{(i)-1}\left(H_{ux}^{(i)}\,\delta x + f_u^{(i)T}\,\delta\lambda\right) \tag{26}$$

$$\delta\dot{x} = A^{(i)}\,\delta x - B^{(i)}\,\delta\lambda \tag{27}$$

$$\delta\dot{\lambda} = -C^{(i)}\,\delta x - A^{(i)T}\,\delta\lambda \tag{28}$$

for $t_{i-1}^+ \leq t \leq t_i^-$, where

$$A^{(i)}(t) = f_x^{(i)} - f_u^{(i)} H_{uu}^{(i)-1} H_{ux}^{(i)} \tag{29}$$

$$B^{(i)}(t) = f_u^{(i)} H_{uu}^{(i)-1} f_u^{(i)T} \tag{30}$$

$$C^{(i)}(t) = H_{xx}^{(i)} - H_{xu}^{(i)} H_{uu}^{(i)-1} H_{ux}^{(i)} \tag{31}$$

Eqs. (18) and (19) suggest that the linearized variational equations have backward sweep solutions of the following forms:

$$\begin{bmatrix} \delta\lambda(t) \\ dy^{(N)} \\ dz^{(N)} \end{bmatrix} = \begin{bmatrix} S^{(N)} & R_1^{(N)} & R_2^{(N)} \\ & Q_{11}^{(N)} & Q_{12}^{(N)} \\ \text{(symm.)} & & Q_{22}^{(N)} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ d\nu^{(N)} \\ dt_N \end{bmatrix} \tag{32}$$

for $t_{N-1}^+ \leq t \leq t_N$, and

$$\begin{bmatrix} \delta\lambda(t) \\ dy^{(i)} \\ dz^{(i)} \\ dp_1^{(i+1)} \\ dp_2^{(i+1)} \end{bmatrix} = \begin{bmatrix} S^{(i)} & R_1^{(i)} & R_2^{(i)} & P_1^{(i)} & P_2^{(i)} \\ & Q_{11}^{(i)} & Q_{12}^{(i)} & b_{11}^{(i)} & b_{12}^{(i)} \\ & & Q_{22}^{(i)} & b_{21}^{(i)} & b_{22}^{(i)} \\ & & & c_{11}^{(i)} & c_{12}^{(i)} \\ \text{(symm.)} & & & & c_{22}^{(i)} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ d\nu^{(i)} \\ dt_i \\ dq_1^{(i+1)} \\ dq_2^{(i+1)} \end{bmatrix} \tag{33}$$

for $t_{i-1}^+ \leq t \leq t_i^-$, $i = 1, \ldots, N-1$, or

$$\begin{bmatrix} \delta\lambda(t) \\ dp_1^{(i)} \\ dp_2^{(i)} \end{bmatrix} = \begin{bmatrix} S_*^{(i)} & R_{*1}^{(i)} & R_{*2}^{(i)} \\ R_{*1}^{(i)T} & Q_{*11}^{(i)} & Q_{*12}^{(i)} \\ R_{*2}^{(i)T} & Q_{*12}^{(i)T} & Q_{*22}^{(i)} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ dq_1^{(i)} \\ dq_2^{(i)} \end{bmatrix} \tag{34}$$

for $t_{i-1}^+ \leq t \leq t_i^-$, $i = 1, \ldots, N$. The differentials $dp^{(i)}$ and $dq^{(i)}$ are defined as follows:

$$dp_1^{(i)} \triangleq \begin{bmatrix} -d\nu^{(N)} \\ \vdots \\ -d\nu^{(i)} \end{bmatrix}, \qquad dq_1^{(i)} \triangleq \begin{bmatrix} dy^{(N)} \\ \vdots \\ dy^{(i)} \end{bmatrix} \tag{35}$$

$$dp_2^{(i)} \triangleq \begin{bmatrix} -dt_N \\ \vdots \\ -dt_i \end{bmatrix}, \qquad dq_2^{(i)} \triangleq \begin{bmatrix} dz^{(N)} \\ \vdots \\ dz^{(i)} \end{bmatrix} \tag{36}$$

The sweep formulations introduced above are generalizations of those in [6] for problems without interior constraints. Speyer has used a sweep formulation which resembles these for solving multi-point boundary value problems and calculating neighboring optimal feedback gains. There are sign errors in his counterpart to Eq. (34).

Consistency of Eqs. (18), (19), and (32-34) yields the following terminal conditions and jump conditions for the sweep quantities:

$$S^{(N)}(t_N) = G_{x_N x_N}^{(N)}, \quad R^{(N)}(t_N) = \left[Y_{x_N}^{(N)T} \;\; z_{x_N}^{(N)T}\right] \tag{37, 38}$$

$$Q^{(N)}(t_N) = \begin{bmatrix} 0 & dY^{(N)}/dt_N \\ (dY^{(N)}/dt_N)^T & dz^{(N)}/dt_N \end{bmatrix} \tag{39}$$

$$S^{(i)}(t_i^-) = G_{x_i x_i}^{(i)} + S_*^{(i+1)}(t_i^+) \tag{40}$$

$$R^{(i)}(t_i^-) = \left[Y_{x_i}^{(i)T} \;\middle|\; (z_{x_i}^{(i)})^T + S_*^{(i+1)}(t_i^+)\,\Delta f_i\right] \tag{41}$$

$$P^{(i)}(t_i^-) = \left[R_{*1}^{(i+1)} \;\middle|\; R_{*2}^{(i+1)}\right]_{t=t_i^+} \tag{42}$$

$$Q^{(i)}(t_i^-) = \left[\begin{array}{c|c} 0 & dY^{(i)}/dt_i \\ \hline (dY^{(i)}/dt_i)^T & dz^{(i)}/dt_i + \Delta f_i^T S_*^{(i+1)}(t_i^+)\Delta f_i \end{array}\right] \tag{43}$$

$$b^{(i)}(t_i^-) = \left[\begin{array}{c|c} 0 & 0 \\ \hline \Delta f_i^T R_{*1}^{(i+1)} & \Delta f_i^T R_{*2}^{(i+1)} \end{array}\right]_{t=t_i^+} \tag{44}$$

$$c^{(i)}(t_i^-) = \left[\begin{array}{c|c} Q_{*11}^{(i+1)} & Q_{*12}^{(i+1)} \\ \hline \text{(Symm.)} & Q_{*22}^{(i+1)} \end{array}\right]_{t=t_i^+} \tag{45}$$

for $i = 1, \ldots, N-1$, where

$$\Delta f_i \triangleq f^{(i)}(t_i^-) - f^{(i+1)}(t_i^+) \tag{46}$$

From Eqs. (32-34), the starred and unstarred matrices are related according to the following expressions:

$$S_*^{(i)} = \underset{\sim}{S}^{(i)}, \quad i = 1, \ldots, N, \tag{47}$$

$$R_*^{(N)} = \left[\underset{\sim}{R}_1^{(N)} \;\middle|\; \underset{\sim}{R}_2^{(N)}\right] \tag{48}$$

$$Q_*^{(N)} = \left[\begin{array}{c|c} \underset{\sim}{Q}_{11}^{(N)} & \underset{\sim}{Q}_{12}^{(N)} \\ \hline \text{(symm).} & \underset{\sim}{Q}_{22}^{(N)} \end{array}\right] \tag{49}$$

$$R_*^{(i)} = \left[\underset{\sim}{P}_1^{(i)} \; \underset{\sim}{R}_1^{(i)} \;\middle|\; \underset{\sim}{P}_2^{(i)} \; \underset{\sim}{R}_2^{(i)} \right], \quad i=1, \ldots, N-1 \tag{50}$$

$$Q_*^{(i)} = \left[\begin{array}{cc|cc} \underset{\sim}{c}_{11}^{(i)} & \underset{\sim}{b}_{11}^{(i)T} & \underset{\sim}{c}_{12}^{(i)} & \underset{\sim}{b}_{21}^{(i)T} \\ & \underset{\sim}{Q}_{11}^{(i)} & \underset{\sim}{b}_{12}^{(i)} & \underset{\sim}{Q}_{12}^{(i)} \\ \hline & & \underset{\sim}{c}_{22}^{(i)} & \underset{\sim}{b}_{22}^{(i)T} \\ (\text{Symm.}) & & & \underset{\sim}{Q}_{22}^{(i)} \end{array} \right], \quad i=1, \ldots, N-1 \tag{51}$$

where, for i = 1, ..., N-1,

$$\underset{\sim}{S}^{(i)} \triangleq S^{(i)} - R^{(i)} Q^{(i)-1} R^{(i)T} \tag{52}$$

$$\underset{\sim}{R}^{(i)} \triangleq R^{(i)} Q^{(i)-1} \tag{53}$$

$$\underset{\sim}{P}^{(i)} \triangleq P^{(i)} - R^{(i)} Q^{(i)-1} b^{(i)} \tag{54}$$

$$\underset{\sim}{Q}^{(i)} \triangleq -Q^{(i)-1} \tag{55}$$

$$\underset{\sim}{b}^{(i)} \triangleq Q^{(i)-1} b^{(i)} \tag{56}$$

$$\underset{\sim}{c}^{(i)} \triangleq c^{(i)} - b^{(i)T} Q^{(i)-1} b^{(i)} \tag{57}$$

Eqs. (52), (53), and (55) hold for i = N also. The remainder are inapplicable.

If Eqs. (27), (28), (32), and (33) are to be consistent, the following differential equations must be satisfied:

$$\dot{S}^{(i)} = -A^{(i)T} S^{(i)} - S^{(i)} A^{(i)} + S^{(i)} B^{(i)} S^{(i)} - C^{(i)} \tag{58}$$

$$\dot{R}^{(i)} = (S^{(i)} B^{(i)} - A^{(i)T}) R^{(i)} \tag{59}$$

$$\dot{P}^{(i)} = (S^{(i)} B^{(i)} - A^{(i)T}) P^{(i)} \tag{60}$$

$$\dot{Q}^{(i)} = R^{(i)T} B^{(i)} R^{(i)} \tag{61}$$

$$\dot{b}^{(i)} = R^{(i)T} B^{(i)} P^{(i)} \tag{62}$$

$$\dot{c}^{(i)} = P^{(i)T} B^{(i)} P^{(i)} \tag{63}$$

$S_*^{(i)}$, $R_*^{(i)}$, and $Q_*^{(i)}$ satisfy differential equations (58), (59), and (61) also.

Of the three sets of backward sweep matrices defined above, the starred set is the most useful. It plays an important role in the second-order optimality conditions described below and may be used to express the neighboring optimum control law (26) in terms of $\delta x(t)$ and $dq_1^{(i)}$.

The Second Variation

The second variation of performance index (4) along a stationary trajectory may be obtained by carefully applying the results stated in [7], for problems with both endpoints variable, but no interior point constraints, to each interval $[t_{i-1}, t_i]$, and then summing over i from 1 to N. The resulting expression is

$$\delta^2 \bar{J} = \sum_{i=1}^{N} \tfrac{1}{2} \left[\delta x_{i-}^T \;\; dt_i \right] \begin{bmatrix} G_{x_i x_i}^{(i)} & (z_{x_i}^{(i)})^T \\ z_{x_i}^{(i)} & dz^{(i)}/dt_i \end{bmatrix} \begin{bmatrix} \delta x_{i-} \\ dt_i \end{bmatrix}$$

$$+ \sum_{i=1}^{N} \tfrac{1}{2} \int_{t_{i-1}}^{t_i} [\delta x^T \delta u^T] \begin{bmatrix} H_{xx}^{(i)} & H_{xu}^{(i)} \\ H_{ux}^{(i)} & H_{uu}^{(i)} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt \, . \tag{64}$$

Speyer [8] has obtained an expression for the second variation which appears to be equivalent to this. The variations in Eq. (64) must, of course, satisfy the linearized versions of Eqs. (1) and (3).

In [6] and [7], sufficient conditions for a weak minimum for two forms of the Bolza problem were derived by adding a complicated expression to the second variation, integrating by parts, and then rearranging terms. The resulting expression was quadratic in a set of independent quantities which fully characterized the set of admissible path variations. A minimally restrictive set of sufficient conditions for a weak minimum was derivable from this expression. An analogous expression for the Bolza problem with interior point constraints is obtained below, by a different approach.

For derivational purposes, additional interior and terminal constraints of dimension $n-r_i$,

$$W^{(i)}(x_i, t_i) - w^{(i)} = 0, \quad i = 1, \ldots, N \tag{65}$$

are introduced. This sequence of new constraints is chosen in such a manner that the matrices

$$\begin{bmatrix} Y_{x_i}^{(i)} \\ W_{x_i}^{(i)} \end{bmatrix} \qquad i = 1, \ldots, N$$

have rank n. $Y_{x_i}^{(i)}$ is assumed to have maximum rank (r_i), so that the original constraints are independent. $w^{(i)}$ and t_i may be thought of as independent parameters which locate the interior and terminal points on their respective manifolds. Small changes in $w^{(i)}$ and t_i, with $y^{(i)}$ held fixed, represent admissible changes in the interior and terminal points. The Lagrange multipliers corresponding to $W^{(i)}$ are denoted by $\mu^{(i)}$. A stationary trajectory for the original problem is characterized by $\mu^{(i)}=0, i=1, \ldots, N$. Changes in $\mu^{(i)}$ may be used to represent violations of corner conditions (10) and transversality condition (12). The quantities $dp_3^{(i)}$ and $dq_3^{(i)}$ are defined as follows:

$$dp_3^{(i)} \triangleq \begin{bmatrix} dw^{(N)} \\ \vdots \\ dw^{(i)} \end{bmatrix}, \qquad dq_3^{(i)} \triangleq \begin{bmatrix} d\mu^{(N)} \\ \vdots \\ d\mu^{(i)} \end{bmatrix} \tag{66}$$

The control perturbation on the ith interval may be written as the sum of its optimal value (26), and the departure, $\delta \hat{u}_D$, from this optimal value:

$$\delta u(t) = -H_{uu}^{(i)-1} (H_{ux}^{(i)} \delta x + f_u^{(i)T} \delta \lambda) + \delta \hat{u}_D(t) \tag{67}$$

The integrand in Eq. (64) then becomes

$$\delta x^T C^{(i)} \delta x + \delta \lambda^T B^{(i)} \delta \lambda + \delta \hat{u}_D^T H_{uu}^{(i)} \delta \hat{u}_D -$$

$$- \delta\lambda^T f_u^{(i)} \delta\hat{u}_D - \delta\hat{u}_D^T f_u^{(i)T} \delta\lambda \triangleq I \tag{68}$$

$\delta\lambda(t)$ and $dp^{(i)}$ are now chosen in terms of $\delta x(t)$ and $dq^{(i)}$ according to Eq. (34), generalized to include $dp_3^{(i)}$ and $dq_3^{(i)}$. It then follows that

$$\delta\dot{x} = A^{(i)}\delta x - B^{(i)}\delta\lambda + f_u^{(i)}\delta\hat{u}_D \tag{69}$$

$$\delta\dot{\lambda} = -C^{(i)}\delta x - A^{(i)T}\delta\lambda + S_*^{(i)} f_u^{(i)}\delta\hat{u}_D \tag{70}$$

$$\frac{d}{dt}(dp^{(i)}) = R_*^{(i)T} f_u^{(i)}\delta\hat{u}_D \tag{71}$$

and

$$I = -\frac{d}{dt}(\delta x^T\delta\lambda) - \frac{d}{dt}(dp^{(i)T} dq^{(i)}) + \delta\hat{u}_D^T H_{uu}^{(i)}\delta\hat{u}_D \tag{72}$$

Eq. (64) may now be manipulated into the form

$$\delta^2\bar{J} = \tfrac{1}{2}[\delta x^T dq_1^{(1)T}]\begin{bmatrix} S_*^{(1)} & R_{*1}^{(1)} \\ R_{*1}^{(1)T} & Q_{*11}^{(1)}\end{bmatrix}\begin{bmatrix}\delta x \\ dq_1^{(1)}\end{bmatrix}_{t=t_o} - \tfrac{1}{2}\sum_{i=1}^{N}[dq_2^{(i)T}\ dq_3^{(i)T}]D^{(i)}(t_i^-)\begin{bmatrix} dq_2^{(i)} \\ dq_3^{(i)}\end{bmatrix} + \tfrac{1}{2}\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\delta u_D^T H_{uu}^{(i)}\delta u_D\, dt \tag{73}$$

where

$$\delta u_D \triangleq \delta u - \delta u^{opt} \tag{74}$$

$$\delta u^{opt}(t) = -H_{uu}^{(i)-1}\left[(H_{ux}^{(i)} + f_u^{(i)T} S_*^{(i)})\ \delta x + f_u^{(i)T} R_{*1}^{(i)} dq_1^{(i)}\right] \tag{75}$$

$$D^{(i)}(t_i^-) \triangleq \xi\ \mathcal{Q}_{22}^{(i)}(t_i^-)\ \xi^T \tag{76}$$

ξ is a column vector involving various backward sweep quantities. Eq. (73) is valid only if condition (a) below is satisfied.

If the initial point and intermediate and terminal manifolds are regarded as fixed, $\delta x(t_o)$ and $dq_1^{(1)}$ may be set equal to zero in Eq. (73). The second variation has now been expressed in terms of quantities which can be varied independently, without violating the linearized state equations and the endpoint and interior constraints. These quantities are $\delta u_D(t)$ (the departure of the control perturbation from its optimal value with $dq_2^{(i)}$ and $dq_3^{(i)} = 0$), and $dq_2^{(i)}$ and $dq_3^{(i)}$ (violations of the transversality and corner conditions). Thus, sufficient conditions for positive definiteness of the second variation (73) are (a) That $S_*^{(i)}$, $R_*^{(i)}$, and $Q_*^{(i)}$ be finite for $t_{i-1}^+ \le t \le t_i^-$, except possibly at t_i^-, $i = 1, \ldots, N$, (b) That $H_{uu}^{(i)}(t)$ be positive-definite for $t_{i-1}^+ \le t \le t_i^-$, $i = 1, \ldots, N$, and (c) That $D^{(i)}$ be negative semidefinite, $i = 1, \ldots, N$.

It follows from Eqs. (59) and (61) that $R_*^{(i)}(t)$ and $Q_*^{(i)}(t)$ can fail to exist only if $S_*^{(i)}(t)$ does not exist [6]. Condition (c) can also be simplified. It is possible to show that the scalar quantity $\mathcal{Q}_{22}^{(i)}(t_i^-)$ is given by

$$\mathcal{Q}_{22}^{(i)}(t_i^-) = \begin{cases} 0, & dY^{(i)}/dt_i \ne 0 \\ -1/\alpha_i, & dY^{(i)}/dt_i = 0 \end{cases} \tag{77}$$

where

$$\alpha_i = \begin{cases} \left[dz^{(i)}/dt_i + \Delta f_i^T S_*^{(i+1)}\Delta f_i\right]_{t=t_i^+}, & i < N \\ dz^{(N)}/dt_N, & i = N \end{cases} \tag{78}$$

Thus, the matrix $D^{(i)}$ is identically zero if $dY^{(i)}/dt_i \ne 0$. If $dY^{(i)}/dt_i = 0$, $D^{(i)}$ has one negative (positive) eigenvalue if α_i is positive (negative), the remaining eigenvalues being zero. Thus condition (c) is satisfied if $dY^{(i)}/dt_i \ne 0$, or if $\alpha_i > 0$ when $dY^{(i)}/dt_i = 0$, $i=1, \ldots, N$.

Sufficient Conditions for a Weak Minimum

Performance index (4) is said to be weakly minimized if it is at a minimum with respect to small admissible perturbations $\delta x(t)$, $\delta u(t)$, and dt_i. Sufficient conditions for a weak minimum are that the first variation vanish and that the second variation be positive-definite. Thus, a set of conditions sufficient to guarantee that performance index (4) has been weakly minimized, subject to state equations (1) and endpoint and interior point constraints (2), (3) may now be stated. Assuming that x(t) is continuous and unbounded, that u(t) is unbounded and continuous except at N-1 corners, $t_1, \ldots, t_{N-1}$, and assuming that $f^{(i)}$, $L^{(i)}$, $Y^{(i)}$, and $g^{(i)}$ are twice continuously differentiable with respect to their arguments, this set of conditions is

(1) That the first-order necessary conditions (1-3) and (8-13) be satisfied,
(2) That $H_{uu}^{(i)}(t)$ be positive-definite for $t_{i-1}^+ \le t \le t_i^-$, $i = 1, \ldots, N$,
(3) That $S_*^{(i)}(t)$ be finite for $t_{i-1}^+ \le t \le t_i^-$, except possibly at t_i^-, $i = 1, \ldots, N$, and
(4) That for $i = 1, \ldots, N$, either
  (4a) $dY^{(i)}/dt_i \ne 0$, or else
  (4b) $\alpha_i > 0$ if $dY^{(i)}/dt_i = 0$.

Condition (3) is a generalized version of the conjugate point and focal point conditions discussed in the classical literature and in [6], [7]. $S_*^{(i)}(t')$ is divergent ($t' < t_i$) if and only if the linearized variational equations have a solution of the form $\delta x(t') = 0$, with $\delta x(t) \not\equiv 0$ on $[t', t_N]$. Such a point, in problems without interior constraints, is known in the classical literature as a conjugate point (relative to the terminal point) if the terminal point is fixed, and a focal point (relative to the terminal manifold), if the terminal point is variable. Arguments against the existence of such points on a minimizing trajectory, except possibly at an endpoint, are valid in the Bolza problem with interior point constraints and discontinuities in the integrand and system equations, as well as in the standard Bolza problem. Thus, if the matrices $S_*^{(i)}(t)$ are definable, $i = 1, \ldots, N$, a necessary condition for a weak minimum is condition (3) above, without the requirement that $S_*^{(1)}(t_o)$ exist.

Condition (4a) is a generalization of the non-tangency condition [1],[6]. Condition (4b) is a convexity condition associated with a variable intermediate time, in cases in which (4a) is not satisfied. If ">0" in (4b) is replaced by "≥0", this condition is a necessary condition for a weak minimum. If condition (4a) is satisfied, this weakened version of (4b) is not necessary.

$S_*^{(i)}(t_i^-)$ does not exist if $Q^{(i)}(t_i^-)$ is singular, which is necessarily the case if $r_i > 1$. If $Q^{(i)}(t)$ is singular throughout the interval $[t_{i-1}^+, t_i^-]$, $S_*^{(i)}(t)$ is not definable on that interval. Lack of definability of $S_*^{(i)}(t)$, for some i, implies that the segment of the stationary trajectory between t_{i-1}^+ and t_i^- is abnormal; that is, it is not imbedded in a family of extremals which satisfy the required constraints at $t_i, \ldots, t_N$. Condition (3) thus includes an implicit normality assumption. The weakened version of condition (3) discussed above is necessary only for normal, nonsingular trajectories. Because of the close correspondence between the necessary conditions and sufficient conditions stated above, it is apparent that the set of sufficient conditions is a minimal set for normal, nonsingular problems. Detailed investigation of abnormal problems would require a more general definition of $\bar{J}$, $H^{(i)}$, and $G^{(i)}$ in Eqs. (5-7).

Condition (3) could be replaced by the condition that $S^{(i)}(t)$ exist for $t_{i-1}^+ \le t \le t_i^-$, $i = 1, \ldots, N$, plus a normality requirement. The resulting set of sufficient conditions is not a minimal set, however, since existence of $S^{(i)}(t)$ for $t_{i-1}^+ \le t \le t_i^-$ is not necessary for a weak minimum. If $r_i \le 1$, $S^{(i)}$ etc. need not be integrated at all, if $dY^{(i)}/dt_i \ne 0$, since $S_*^{(i)}(t_i^-)$ can then be evaluated directly.

Only weak variations have been considered above. Strong variations require that $\delta x(t)$ be small, without requiring that $\delta u(t)$ be small. A necessary condition for a strong relative minimum is that u(t) minimize $H^{(i)}[x(t), u(t), \lambda(t), t]$ globally, rather than just locally, on $[t_o, t_N]$ (Weierstrass condition). Sets of sufficient conditions for a strong minimum include the sufficient conditions stated above and a strengthened version of the Weierstrass condition.

In summary, implementation of the sets of necessary and sufficient conditions discussed above involves the following steps (1) Determine a trajectory which satisfies the first-order necessary conditions. (2) Verify that condition (2) is satisfied and check condition (4a), $i = 1, \ldots, N$. (3) Repeat the following process for $i = N, \ldots, 1$: (a) If condition (4a) is not satisfied, verify that condition (4b) is satisfied. (b) Evaluate $S^{(i)}(t_i^-)$, $R^{(i)}(t_i^-)$, $Q^{(i)}(t_i^-)$ using Eqs. (37-39) or (40), (41), and (43). (c) If $r_i = 0$, or if $r_i = 1$ and condition (4a) is satisfied, let $t_i' = t_i^-$ and proceed to step (e); otherwise, (d) Integrate $S^{(i)}(t)$, $R^{(i)}(t)$, and $Q^{(i)}(t)$ backward, using Eqs. (58), (59), and (61), from t_i^- to some time $t_i' \ge t_{i-1}^+$ at which $Q^{(i)}(t)$ is sufficiently nonsingular to be inverted accurately. (e) Evaluate $S_*^{(i)}(t_i')$ using Eqs. (47) and (52). (f) Integrate $S_*^{(i)}(t)$ backward from t_i' to t_{i-1}^+, using Eq. (58).

If the optimality conditions described above are satisfied, the perturbation feedback control law (75) is an optimal control law, and the coefficients of $\delta x(t)$ and $dq_1^{(i)}$ are optimal feedback gains. Changes in the intermediate and terminal times due to such perturbations may be calculated using Eq. (34). Changes in state at the intermediate and terminal points may be determined through an analogous expression for $dp_3^{(i)}$ in terms of $\delta x(t)$ and $dq_1^{(i)}$, if the backward sweep is modified to include $dp_3^{(i)}$ and $dq_3^{(i)}$, as was done in the derivation above. The appropriate differential equations and boundary conditions are readily obtained.

For the standard Bolza problem, Brusch and Vincent [11] have described a second-order optimality condition which has been shown to be necessarily satisfied if the conditions in [6] are satisfied. A generalized form of their condition, for problems with interior point constraints, would imply that small violations of the corner and transversality conditions (or equivalently, intersecting the terminal and interior manifolds at different points) cannot possibly improve the performance index. This condition is clearly satisfied if conditions (1)-(4) above are satisfied.

It has been assumed above that the initial point is fully specified. If this is not the case, the results of this paper may be combined with those in [7] to yield backward, forward, and forward/backward sweep sets of optimality conditions.

If t_i is specified, transversality condition (11) or (13) is removed from the first-order necessary conditions. The differentials dt_i and $dz^{(i)}$ are removed from $dp_2^{(i)}$ and $dq_2^{(i)}$. If $i = N$, the boundary condition on $S^{(N)}$ remains the same, but those on $R^{(N)}$ and $Q^{(N)}$ become

$$R^{(N)}(t_N) = Y_{x_N}^{(N)}, \quad Q^{(N)}(t_N) = 0 \tag{79}$$

If $i < N$, the boundary conditions on $S^{(i)}$, $P^{(i)}$, and $c^{(i)}$ remain the same, but those on $R^{(i)}$, $Q^{(i)}$, and $b^{(i)}$ become

$$R^{(i)}(t_i^-) = Y_{x_i}^{(i)T}, \quad Q^{(i)}(t_i^-) = 0, \quad b^{(i)}(t_i^-) = 0 \tag{80}$$

Conditions (4a) and (4b) are inapplicable for each i at which t_i is specified.

Problems with Natural Corners

If H has a double minimum at time t_i, the optimal control may be discontinuous, even in the absence of path constraints. Such a point is called a natural corner. A natural corner may be represented as an interior point constraint of dimension zero at an unspecified time t_i, t_i being determined implicitly by corner condition (11), with $G^{(i)}$ and its derivatives equal to zero. Eqs. (40), (41), and (43) reduce to

$$S^{(i)}(t_i^-) = S_*^{(i+1)}(t_i^+) \tag{81}$$

$$R^{(i)}(t_i^-) = (z_{x_i}^{(i)})^T + S_*^{(i+1)}(t_i^+)\,\Delta f_i \tag{82}$$

$$Q^{(i)}(t_i^-) = \alpha_i \tag{83}$$

where $$z_{x_i}^{(i)} = H_x(t_i^-) - H_x(t_i^+) \tag{84}$$

$$dz^{(i)}/dt_i = [H_t + H_x f]_{t=t_i^-} - [H_t + H_x f]_{t=t_i^+} - 2\Delta f_i^T H_x^T (t_i^+) \tag{85}$$

α_i is given by Eq. (78). The functions $L^{(i)}$ and $f^{(i)}$ are assumed to be the same for all i.

Necessary and sufficient conditions for a weak minimum are those given above for problems with interior point constraints, with jump conditions (40), (41), and (43) replaced by Eqs. (81-83). Nontangency condition (4a) is inapplicable. Condition (4b) should be checked instead.

For problems with $L \equiv 0$, Moyer [12] defines a wavefront in state space as the locus of endpoints of all extremals which originate at a fixed point and terminate at a fixed final time. λ is orthogonal to this wavefront. He distinguishes between natural corners which are reflective and refractive in nature, according to whether or not an extremal crosses the wavefront at a corner, and argues that extremals passing through reflection points are nonoptimal. It seems likely that reflection points are ruled out by condition (4b).

Control Constrained Problems

If the s-dimensional constraint

$$h(u,t) \le 0 \tag{86}$$

is added to the problem definition, as given in Eqs. (1-4), first-order conditions for a local minimum are Eqs. (2-4), (8-13), (86) and

$$\eta_j \begin{cases} = 0, & h_j < 0 \\ \ge 0, & h_j = 0 \end{cases} \qquad j = 1, \ldots, s \tag{87}$$

where the Hamiltonian is now defined as

$$H(x,u,\lambda,t,\eta) \triangleq L + \lambda^T f + \eta^T h \tag{88}$$

η is a vector of undetermined multipliers. Interior point constraints of the form given by Eq. (3), i = 1, ..., N-1, are assumed to be absent. Instead the times t_i, i = 1, ..., N-1, may be used to denote times at which one or more control constraints become effective or ineffective. Denoting by $h^{(i)}$ the vector of effective constraints between t_{i-1}^+ and t_i^-, and by $\eta^{(i)}$ the corresponding multiplier vector, it can be shown that δx, $\delta\lambda$, and δu are still determined according to Eqs. (26-28), with $H_{uu}^{(i)-1}$ replaced by $E^{(i)}$ in Eqs. (26) and (29-31), where

$$M^{(i)} = \begin{bmatrix} E^{(i)} & F^{(i)T} \\ F^{(i)} & G^{(i)} \end{bmatrix} \triangleq \begin{bmatrix} H_{uu} & h_u^{(i)T} \\ h_u^{(i)} & 0 \end{bmatrix}^{-1} \tag{89}$$

The partitions $E^{(i)}$ and $F^{(i)}$ of $M^{(i)}$ correspond in dimension to H_{uu} and $h_u^{(i)}$. $\delta\eta^{(i)}(t)$ is given by

$$\delta\eta^{(i)}(t) = -F^{(i)}\left(H_{ux}\,\delta x + f_u^T\,\delta\lambda\right) \tag{90}$$

With these modifications, the sweep results obtained for problems with interior point constraints carry over to problems with control variable inequality constraints. Since $r_i = 0$, i = 1, ..., N-1, jump conditions (40), (41), and (43) assume the form of Eqs. (81-83). Eqs. (64) and (73) for the second variation are still valid (with $G_{x_i x_i}^{(i)} = 0$, i = 1, ..., N-1), except that only control variations which are consistent with

$$h_u^{(i)}\delta u = 0, \tag{91}$$

as well as Eq. (14) and the second row of Eq. (19), are considered. Control variations which violate Eq. (91) either violate control constraints (86) or else produce a first-order increase in J (except in the immediate vicinity of a switch point). Thus, the family of neighboring extremals under consideration is defined here in a more narrow sense in problems with control constraints than in problems without them. A weak neighborhood of a stationary trajectory includes only trajectories having the same sequence of effective and ineffective constraints. Changes in state and switching times are small, and changes in control are small except possibly near switch points.

If the optimal control is discontinuous at t_i, i = 1, ..., N-1, but continuous elsewhere, and $\dot{x}(t)$ is continuous and unbounded, and f, L, h, $Y^{(N)}$, and $g^{(N)}$ are twice continuously differentiable with respect to their arguments, a set of sufficient conditions for a weak minimum is

(1) That the first-order necessary conditions (2-4), (8-13), and (86) be satisfied and that Eq. (87) be satisfied, with the equality in the second line holding only at switch points,
(2) That $M^{(i)}(t)$ exist and that $\delta u^T H_{uu}\,\delta u$ be positive for all $\delta u \neq 0$ such that $h_u^{(i)}\delta u = 0$, $t_{i-1}^+ \le t \le t_i$, i = 1, ..., N,
(3) That $S_*^{(i)}$ be finite for $t_{i-1}^+ \le t \le t_i^-$, i = 1, ..., N, except possibly at t_N, and
(4) That α_i be greater than zero, i = 1, ..., N-1, and either $\alpha_N > 0$ or else $dY^{(N)}/dt_N = 0$.

A necessary condition for a strong relative minimum is that the control minimize H globally, consistent with the control constraint, for $t_o \le t \le t_N$. A set of sufficient conditions for a strong minimum includes a strengthened form of the latter condition, plus the sufficient conditions given above for a weak minimum.

Jacobson and Mayne [9] and Dyer and McReynolds [10] have proposed second-order algorithms, based on a backward sweep, for the determination of stationary trajectories in problems with control constraints. Dyer and McReynolds observe that convexity condition (4), in weakened form, is necessary for local optimality and cite an analytic example in which a stationary trajectory is nonoptimal because it fails to satisfy this condition. Jacobson and Mayne state conditions sufficient for reduction in the performance index and the terminal error from one iteration to the next. These conditions are closely related to sufficient conditions for local optimality.

The results of this paper are applicable to a variety of problems with control constraints, as long as the control is discontinuous at t_i, i=1, ..., N-1. Jacobson and Mayne consider the restricted class of such problems in which all components of the control are bounded above and below by constants and H is linear in u. In this case, Riccati equation (58) degenerates into a linear differential equation. Hence, condition (3) can be ignored, since the sweep variables cannot diverge in finite time if $A^{(i)}$, $B^{(i)}$, and $C^{(i)}$ are well behaved and condition (4) is satisfied. Condition (2) can also be ignored.

If there are no constraints on the terminal state, and t_N is given, the results of this paper reduce directly to the corresponding results of Jacobson and Mayne, if the control is bounded as described above. (There is a typographical error in their counterpart to Eq. (85).) The conditions stated above are similarly applicable if t_N is unspecified and a single terminal constraint is given.

If there are enough terminal constraints that $Q^{(N)}(t_N)$ is singular, however, the conditions stated above break down, since $Q^{(N)}(t)$ must be constant, and hence identically singular. The trajectory between t_{N-1} and t_N is thus abnormal. If t_N is given and there are r terminal constraints, the trajectory is abnormal at least as far back as t_{N-r} (back to t_{N-r+1} if t_N is free). The case with t_N specified is studied in greater detail by Jacobson and Mayne.

If the control is continuous at t_i, modifications in the backward sweep are required. The jump relations become simply

$$S_*^{(i)}(t_i^-) = S_*^{(i+1)}(t_i^+), \quad R_*^{(i)}(t_i^-) = R_*^{(i+1)}(t_i^+) \qquad (92)(93)$$

$$Q_*^{(i)}(t_i^-) = Q_*^{(i+1)}(t_i^+) \qquad (94)$$

With these modifications, the optimality conditions stated above for problems with discontinuous control can be applied to problems with continuous control.

Jacobson and Mayne consider problems of this type also. The sufficient conditions given above are less restrictive than those of Jacobson and Mayne. In place of condition (2) above, they require that H_{uu} be positive-definite. This is not necessary. In addition, in problems with terminal constraints, they require that the linearized system be completely controllable. This is also unnecessary. All that is required is that the linearized system be controllable in the reduced sense [13], that is, controllable to any neighboring terminal manifold, rather than to any neighboring terminal point. Controllability in the reduced sense is equivalent to normality [13], which is already taken into account through the condition on the existence of $S_*^{(i)}(t)$. Finally, the requirement that their set of backward sweep matrices be finite between t_o and t_N is unnecessarily restrictive in problems with terminal constraints. Their second partial derivative of the return function (taken with $\nu^{(N)}$ held constant) is not the same as $S_*^{(i)}$. Finiteness of the latter (except at t_f) guarantees finiteness of the former, but the converse is not necessarily true.

Conclusions

A minimal set of sufficient conditions for a weak minimum has been derived for the nonsingular Bolza problem of variational calculus, with interior point constraints and discontinuities in the system equations and integrand. This set of conditions includes generalized versions of the conjugate point/focal point, normality, convexity, and non-tangency conditions associated with the standard Bolza problem. Slightly weakened versions of these conditions are found to be necessary for a weak minimum for normal nonsingular problems. These conditions are much easier to apply than the classical conditions of Bliss [1] and Denbow [2]. Feedback gains on deviations in the state variables and interior and terminal constraints, for use in a neighboring optimum control law and for predicting changes in the intermediate and terminal states and times, are easily calculated while the second-order optimality conditions are being checked.

Second-order optimality conditions for problems with natural corners are deduced. Convexity condition (4b) appears to be related to Moyer's condition concerning reflection points [12]. Analogous optimality conditions are obtained for control constrained problems. These conditions reduce to those in [9] in some cases, but are less restrictive than those in [9] in other cases.

References

[1] Bliss, G. A., Lectures on the Calculus of Variations, Univ. of Chicago Press, 1946.

[2] Denbow, C. H., "A Generalized Form of the Problem of Bolza," Contributions to the Calculus of Variations, Univ. of Chicago Press, 1933-1937, pp. 453-484.

[3] Denham, W. F., "Steepest Ascent Solution of Optimal Programming Problems," Ph. D. thesis, Harvard Univ., 1963.

[4] Bryson, A. E., Jr. and Ho, Y. C., Applied Optimal Control, Blaisdell, 1969.

[5] McReynolds, S. R., "A Successive Sweep Method for Solving Optimal Programming Problems," Ph.D. thesis, Harvard Univ., 1965.

[6] Wood, L. J. and Bryson, A. E., Jr., "Second-Order Optimality Conditions for Variable End Time Terminal Control Problems," AIAA Journal, 11, 9, Sept. 1973, pp. 1241-1246.

[7] Wood, L. J., "Second-Order Optimality Conditions for the Bolza Problem with Variable End Points and Separated End Conditions," 1973 JACC, Columbus, Ohio.

[8] Speyer, J. L., "Optimization and Control of Nonlinear Systems with Inflight Constraints," Ph. D. thesis, Harvard Univ., 1968.

[9] Jacobson, D. H. and Mayne, D. Q., Differential Dynamic Programming, Elsevier, 1970.

[10] Dyer, P. and McReynolds, S. R., The Computation and Theory of Optimal Control, Academic Press, 1970.

[11] Brusch, R. G. and Vincent, T. L., "Numerical Implementation of a Second-Order Variational Endpoint Condition," AIAA Journal, 8 No. 12, Dec. 1970, pp. 2230-2235.

[12] Moyer, H. G., "Optimal Control Problems that Test for Envelope Contacts," Journal of Optimization Theory and Applications, 6, No. 4, 1970, pp. 287-298.

[13] Schmitendorf, W. E., and Citron, S. J., "On the Applicability of the Sweep Method to Optimal Control Problems," IEEE Transactions on Automatic Control, v. AC-14, No. 1, Feb. 1969, pp. 69-72.